Note on Static Game with Complete Information

Zepeng CHEN
The HK PolyU

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1 Strategy, Information and Game

1.1 Game and Assumptions

Definition 1.1 (Common knowledge vs. Mutual knowledge)

1. *A property Y is said to be* ***mutual knowledge*** *if all players know Y (but don't necessarily know that others know it).*
2. *A property Y is* ***common knowledge*** *if everyone knows Y, everyone knows that everyone knows Y, everyone knows that everyone know that everyone knows Y, ..., ad infinitum. Clearly, common knowledge implies mutual knowledge but not vice-versa.*

Definition 1.2 (Normal (Strategic) Form of Game)

1. *Player Set, e.g.* $N = \{1, 2, 3, 4, \cdots, n\}$.
2. *Set of strategy profiles, e.g.* $\prod_{j \in N} S_j$, *where* S_j *is the strategy set for player* j.
3. *Payoff function for each player:* $U_i(s_1, s_2, \ldots, s_n)$.

1.2 Rationality, Dominance and Best Response

Definition 1.3 (Set of k-rationalizable strategies (Li, 2022))

Set $\Sigma_i^0 = \Sigma_i$, *and recursively define*

$\Sigma_i^n = \{\sigma_i \in \Sigma_i^{n-1} : \exists \sigma_{-i} \in \times_{j \neq i} \left(\hat{\Sigma}_j^{n-1}\right) \text{ s.t.} U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) \, \forall \sigma_i' \in \Sigma_i^{n-1}\}$,

where $\hat{\Sigma}_j^{n-1}$ *is the convex hull of* Σ_j^{n-1}.

Note on Convex hull *Here convex full can capture mixed strategies.*

Note on Interpretation (Kartik, 2009) *Here* $\sigma_i \in \sum_i$ *is a k-rationalizable strategy* ($k \geq 2$)*for player* i *if it is a best response to some strategy profile* $\sigma_{-i} \in \sum_{-i}$ *such that each* σ_i *is (k-1)-rationalizable for player* $j \neq i$.

Definition 1.4 (Set of rationalizable strategies (Li, 2022))

The set of rationalizable strategies for player i is

$$R_i = \cap_{n=0}^{\infty} \Sigma_i^n.$$

Note on Interpretation (Kartik, 2009) *Here $\sigma_i \in \sum_i$ is rationalizable for player i if it is k-rationalizable for all $k \geq 1$.*

Definition 1.5 (Rational player and Rationalizable strategy profile (Li, 2022))

1. *A rational player will not play a strategy that is never a best response.*
2. *The set of strategies that survive iterated elimination of strategies that are never a best response are rationalizable.*
3. *A strategy profile is rationalizable if the strategy prescribed for each player is rationalizable.*

Theorem 1.1

The set of rationalizable strategies R_i for each player i is nonempty and contains at least one pure strategy. Further each $\sigma_i \in R_i$ is a best response to an element of $\times_{j \neq i}$ convex hull (R_j).

Note on Nonempty *Each period you compare all strategies to find best responses, you can always find at least one strategy.*

Definition 1.6 (Strict and Weak Dominance)

1. *A strategy s_i is strictly dominated for player i if there exists $\sigma_i' \in \sum_i$ such that*

$$u_i(s_i, s_{-i}) < u_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

2. *A strategy s_i is weakly dominated by a_i' if $u_i(s_i, s_{-i}) \leq u_i(\sigma_i', s_{-i})$ for $\forall s_{-i} \in S_{-i}$, and the inequality is strict for some s_{-i}.*

Definition 1.7 (Best response mapping / correspondence)

Given an n-player game, player i's best response (mapping: one to multiple) to the strategies x_{-i} of the other players is the strategy x_i^* that maximizes players i's payoff $\pi_i(x_i, x_{-i})$:

$$x_i^*(x_{-i}) = \arg\max_{x_i} \pi_i(x_i, x_{-i}).$$

When the best response is not unique, we use the definition of correspondence, i.e. $n \rightarrow n$.

Note on BR *Best response may not be unique.*

Lemma 1.1 (Derivative of BR)

Given payoff $u_i(q_i, x)$ and best response $r_i(q_j) : \frac{\partial u_i(r(q_j), q_j)}{\partial q_i} = 0$, then the derivative is

$$\frac{\partial^2 u_i}{\partial q_i^2}\frac{\partial r}{\partial q_j} + \frac{\partial^2 u_i}{\partial q_i \partial q_j} = 0 \to \frac{dr}{dq_j} = -\frac{\partial^2 u_i}{\partial q_i \partial q_j} / \frac{\partial^2 u_i}{\partial q_i^2}$$

1.3 From Pure Strategy to Mixed Strategy

Note that in business area, we focus on pure strategy applications. Thus, we omit further discussions of mixed strategy in later sections.

Definition 1.8 (Pure or Mixed Strategy)

1. *A pure strategy is the action which a player chooses for sure.*
2. *A mixed strategy σ_i of player i is a probability distribution over pure strategies. The set of mixed strategies is denoted by Σ_i ($\Sigma \equiv \times_{i \in N}\Sigma_i$). The expected payoff is*

$$U_i(\sigma) = \sum_{a \in A} \left(\Pi_{j=1}^{N} \sigma_j(a_j)\right) u_i(a) \tag{1}$$

$$= \sum_{a_i \in A_i} [\sigma_i(a_i) U_i(a_i, \sigma_{-i})] \tag{2}$$

Note on Mixed strategy *Here randomization is independent, and we relax this assumption in correlated equilibrium.*

Note on Expected payoff (1) *captures the expected payoff via summing all weighted expected payoff given strategy profiles.* (2) *captures the expected payoff via summing all weighted expected payoff given strategies with nonnegative probability.*

2 Solution Concept 1: Minmax Theorem in Strictly Competitive Games (Felix, 2022, Lec. 8)

Definition 2.1 (Strictly competitive game)

A two-player, strictly competitive game is a two-player game with the property that, for every two strategy profiles s and s',

$$u_1(s) \geq u_1(s') \text{ and } u_2(s) \leq u_2(s')$$

Note on Special case *Constant-sum games are special cases of strictly competitive games, and zero-sum games are special cases of constant-sum games.*

Definition 2.2 (Maxmin strategy and Maxmin value)

The maxmin strategy for player i is a strategy that maximize i's worst-case payoff

$$\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$$

and the maxmin value for player i is the payoff guaranteed by the maxmin strategy.

$$\max_{s_i} \min_{s_{-i}} u_i\left(s_1, s_2\right)$$

Definition 2.3 (Minmax strategy and Minmax value)

The minmax strategy for player i is a strategy that minimizes the other player $-i$'s best-case payoff

$$\arg\min_{s_i} \max_{s_{-i}} u_{-i}\left(s_1, s_2\right)$$

and the minmax value for player i is the payoff achieved by the minmax strategy.

$$\min_{s_i} \max_{s_{-i}} u_{-i}\left(s_1, s_2\right)$$

Theorem 2.1 (Minmax theorem (von Neumann))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

Remark In geometry, it means a saddle point.

3 Solution Concept 2: Iterated Strict Dominance Equilibrium

These strategy profiles are usually derived via iterated elimination of strictly dominated strategies (IESDS).

Definition 3.1 (iterated strict dominance equilibrium)

Given strong rationality and common knowledge assumption, repeatedly applying strict dominance to derive this equilibrium.

Note on Uniqueness *If we do elimination of strictly dominated strategies, then the equilibrium is unique, otherwise the uniqueness can not be ensured.*

Note on Condition *Some (very few though) games can be "solved" by iterated elimination of strictly dominated strategies.*

Note on Order independence *An important feature of iterated elimination of strictly dominated strategies is that the order of the elimination has no effect on the set of strategies that remain in the end.*

Note on Weakly dominated *A strategy that is weakly dominated cannot be ruled out based only on principles of rationality. Iterated elimination of weakly dominated strategies is order dependent, for example,*

	L	C	R
T	$1,1$	$1,1$	$0,0$
B	$0,0$	$1,2$	$1,2$

.

Note on Mixed strategy in IESDS

1. *If there is a mixed strategy σ_i' dominate all pure strategies, then it dominate all mixed strategies.*
    - *Since mixed strategy is convex combination of pure strategy.*
2. *A pure strategy may be dominated by a mixed strategy, and vice versa.*
    - *This property is very useful to prove some mixed strategies is dominated.*
    - *For example, 0.5T+0.5M is dominated by B, while T and M are not dominated by B. Using this property, we con show that all mixed strategies of T and M is dominated, by the new strategy we construct. For example, for 0.9T+0.1M=0.8T+0.2*(0.5T+0.5M), we does know whether the former is dominated, but the latter is dominated by 0.8T+0.2*B.*

	L	R
T	3	0
M	0	3
B	2	2

    - *For example, there are no strictly dominated pure strategies for player 1 nor for player 2. However, if player 1 mixes between B (with prob q) and C (with prob 1-q), he obtains an expected utility that exceeds that from selecting F. Thus we can eliminate F from the matrix, since it is strictly dominated by a randomization between B and C.*

	F	C	B
F	$0,5$	$2,3$	$2,3$
$\mathrm{C}(1-\mathrm{q})$	$2,3$	$0,5$	$3,2$
$\mathrm{B}(\mathrm{q})$	$5,0$	$3,2$	$2,3$

.

3. *A mixed strategy that assigns positive probability to a (strictly) dominated pure strategy must be (strictly) dominated.*
    - *This property is very useful to eliminate dominated strategies and get a smaller game.*

Proposition 3.1 (iterated dominance equilibrium and NE (Gibbons, 1992, p. 12))

1. *In the n-player normal-form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, ..., s_n^*)$, then these strategies are the unique Nash equilibrium of the game.*
2. *In the n-player normal-form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, if the strategies $(s_1^*, ..., s_n^*)$ are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.*

Note on *Note that NE is stronger than IDSDS, actually, if we cannot eliminate strictly dominated strategies, all strategy profiles survive the application of IDSDS (Felix, 2022, Lec. 3).*

Proof We prove (2) by contradiction, assuming that $(s_1^*, ..., s_n^*)$ is a NE, and there exists s_i^* such that s_i^* is strictly dominated. Then it means s_i^* is not best response and contradicts the definition of NE. This proof actually shows that NE is stronger than iterated strict dominance equilibrium, that is, every NE survives iterated elimination of strictly dominated strategies. (1) can be proved by contradiction too. ■

Lemma 3.1

Rationalizability and iterated strict dominance coincide in two-player games.

Proof This is equivalent to show that

$$\sigma_i \text{ is strictly dominated } \iff \sigma_i \text{ is never a best response.}$$

■

Note on *With more than two players, the equivalence between being strictly dominated and never a best response breaks down. The reason is that mixed strategies require that player's randomizations are independent. For example, D is not a best response to player 3, while it is also not dominated.*

	L	R
U	3	0
D	0	0

Figure 1: A

	L	R
U	0	3
D	3	0

Figure 2: B

	L	R
U	0	0
D	0	3

Figure 3: C

	L	R
U	2	0
D	0	2

Figure 4: D

4 Solution Concept 3: Nash Equilibrium

4.1 Nash Equilibrium

Definition 4.1 (Pure vs Mixed Nash Equilibrium)

A mixed-strategy profile σ^* is a Nash equilibrium if, for all players i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall s_i \in S_i.$$

A pure-strategy Nash equilibrium is a pure-strategy profile that satisfies the same conditions.

Note on Pure strategy equilibria in OM papers *Pure strategy equilibria is attractive in OM papers for two reasons. First, the simplicity of their structure makes it possible to link them to the actual behavior of the competing firms. Second, no firm has ex post regret after observing* *the choice of the other firm. (Tirole, 1988)*

Lemma 4.1 (NE as mutual best response)

An outcome $(x_1^*, ...x_n^*)$ is a Nash equilibrium of the game if x_i^* is a best response to x_{-i}^* for all $i = 1, ..n$.

Note on Computation *This lemma can be used to find NE. For example in Matching pennies.*

$$\begin{array}{cccc} & & \multicolumn{2}{c}{2} \\ & & \text{H} & \text{T} \\ 1 & \text{H} & -1, 1 & 1, -1 \\ & \text{T} & 1, -1 & -1, 1 \end{array}$$

Suppose player 1 chooses H with probability r and T with $1 - r$, player 2 chooses H with q and T with $1 - q$. Then player 1's expected utility is $u_1 = -(2q - 1)(2r - 1)$. And FOC implies that $\frac{\partial U_1}{\partial r} = -2(2q - 1)$: > 0 if $q < \frac{1}{2}$, $= 0$ if $q = \frac{1}{2}$, < 0 if $q > \frac{1}{2}$. Thus the best response correspondence for player 1 and 2 are

$$\beta_1(q) = \begin{cases} 1 & q < \frac{1}{2} \\ [0, 1] & q = \frac{1}{2} \\ 0 & q > \frac{1}{2} \end{cases} \qquad \beta_2(r) = \begin{cases} 0 & r < \frac{1}{2} \\ [0, 1] & r = \frac{1}{2} \\ 1 & r > \frac{1}{2} \end{cases}$$

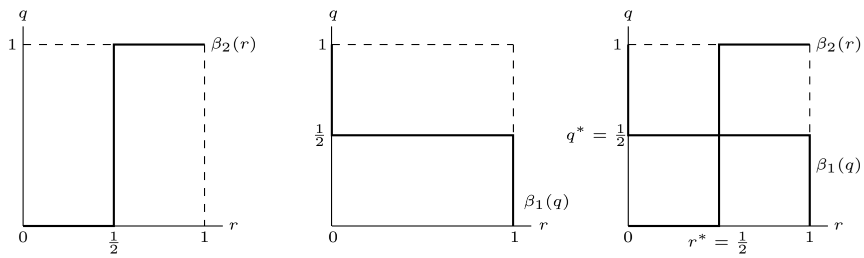

Figure 5: MNE for Matching Pennies

Note on Mixed NE

1. *A pure-strategy NE is a degenerate mixed-strategy NE, and there may exist no pure-strategy NE. For example, there is only a unique mixed NE where both chooses Head with probability 1/2.*

	Head	*Tail*
Head	$\underline{1}, -1$	$-1, \underline{1}$
Tail	$-1, \underline{1}$	$\underline{1}, -1$

2. *Mixed strategy means to randomize to confuse your opponent.*
3. *Mixed strategies are a concise description of what might happen in repeated (infinite) play.*
4. *Dominated strategies are never used in mixed Nash equilibria, even if they are dominated by another mixed strategy.*

Note on Infinite mixed NE *There can be infinite mixed NE. For example, this game has two pure NEs: (In, Accept) and (Out, Fight). When player 1 chooses Out, 2 is indifferent between two strategies. Thus there is a continuum of mixed NE: player 1 chooses "Out", while 2 chooses*

"Fight" with a probability at least 1/3. Note that the pure NE (Out, Fight) belongs to this family of mixed NE.

	Accept	Fight
Out	$0, 4$	$\underline{0}, \underline{4}$
In	$\underline{1}, \underline{2}$	$-2, 1$

Note on Condition *In a Nash equilibrium,*

1. *Each player's strategy is required to be optimal given his belief about other players' strategies.*
2. *Each player's belief about other players' strategies is required to be correct.*

The latter requirement is strong and somewhat unreasonable.

Note on Interpretation of mixed NE *Mixed strategies can be viewed as pure strategies in a perturbed game, and a mixed NE can be viewed as a steady state. A mixed strategy profile can be viewed as players' beliefs about each other's strategic choices.*

Note on Rationlizable in mixed NE (Li, 2022) *Every strategy used with positive probability in some mixed strategy NE is rationalizable.*

Lemma 4.2 (Property of Mixed NE)

In a mixed-strategy NE, each player is indifferent among all those pure strategies that he chooses with positive probability, that is, $E[u_i(s_i, p_{-i})]$ must be the same for all such strategies.

Note on Interpretation and Proof *This property can be directly seen and proved from Equation 2.*

Note on Condition and Computation and Geometry Interpretation *This property is very useful to compute and find a mixed NE, however, we should pay attention to its condition of* ***positive probability****. For those with zero probability, we do not require their expected payoff to be the same. For example, if we denote the probability of L, M and R as q_1, q_2 and $1 - q_1 - q_2$, and compute mixed NE by this property, then there is no mixed NE. The correct way to find mixed NE is next verify the combinations $\{L, M\}$, $\{L, R\}$ and $\{M, R\}$.*

	L	M	R
T	$3, 4$	$1, 3$	$3, 0$
B	$0, 1$	$2, 3$	$0, 4$

Lemma 4.3 (Oddness Theorem (Wilson 1971))

Almost *every finite game has an odd number of NE's in mixed strategies.*

Note on Almost *Actually, there is a counter example. This game has two pure NEs: (T,L) and (B,R), but no mixed NE.*

	L	R
T	$1,1$	$0,0$
B	$0,0$	$0,0$

Lemma 4.4 (Fixed points and NE)

Nash equilibrium must satisfy $\frac{\partial \pi_i}{\partial x_i} = 0$ *for all player* i, *let* $f_i(x_1, ..x_n) = \frac{\partial \pi_i}{\partial x_i} + x_i$, *then we can find NE by fixed points theorem.*

$$f_i(x_1^*, ..x_n^*) = x_i^* \rightarrow \frac{\partial \pi_i(x_1^*, ...x_n^*)}{\partial x_i} = 0, \quad \forall i$$

4.2 Existence of NE

Theorem 4.1 (Nash's Theorem)

Every finite strategic-form game has a mixed-strategy equilibrium.

Note on Condition *If the game is not finite, then the existence can not be guaranteed.*

Proof P10 Li duozhe's note. ■

Theorem 4.2 (Debreu 1952)

Consider a strategic-form game whose strategy spaces S_i *are nonempty compact*[a] *convex subsets of an Euclidean space. If the payoff functions* u_i *are continuous in* s *and quasi-concave in* s_i, *there exists a pure-strategy Nash equilibrium.*

[a] Strategy space is compact if it is closed and bounded

Corollary 4.1

Suppose that a game is symmetric, and for each player, the strategy space is compact and convex and the payoff function is continuous and quasiconcave with respect to each player's own strategy. Then, there exists at least one symmetric pure strategy NE in the game.

Theorem 4.3 (Glicksberg 1952)

Consider a strategic-form game whose strategy spaces S_i *are nonempty compact subsets of a metric space. If the payoff functions* u_i *are continuous then there exists a Nash equilibrium in mixed strategies.*

Note on Comparison *Compared to Theorem 4.1, this theorem generalizes the condition to infinite strategy space. Compared to Theorem 4.2, this theorem relaxes the condition of quasi-concavity, however, this theorem does not guarantee that the NE is in pure strategies.*

4.3 Uniqueness of NE

4.3.1 Contraction Mapping Argument

Definition 4.2 (Contraction Mapping Argument)

Mapping $f(x) : R^n \to R^n$ *is a contraction iff* $||f(x_1) - f(x_2)|| \leq \alpha||x_1 - x_2|| \quad \forall x_1, x_2, \alpha < 1$.

For example, in Figure 6, contraction mapping can derive a converged iterated series, which satisfies $|f'(x)| < 1$; otherwise, the series will not converge.

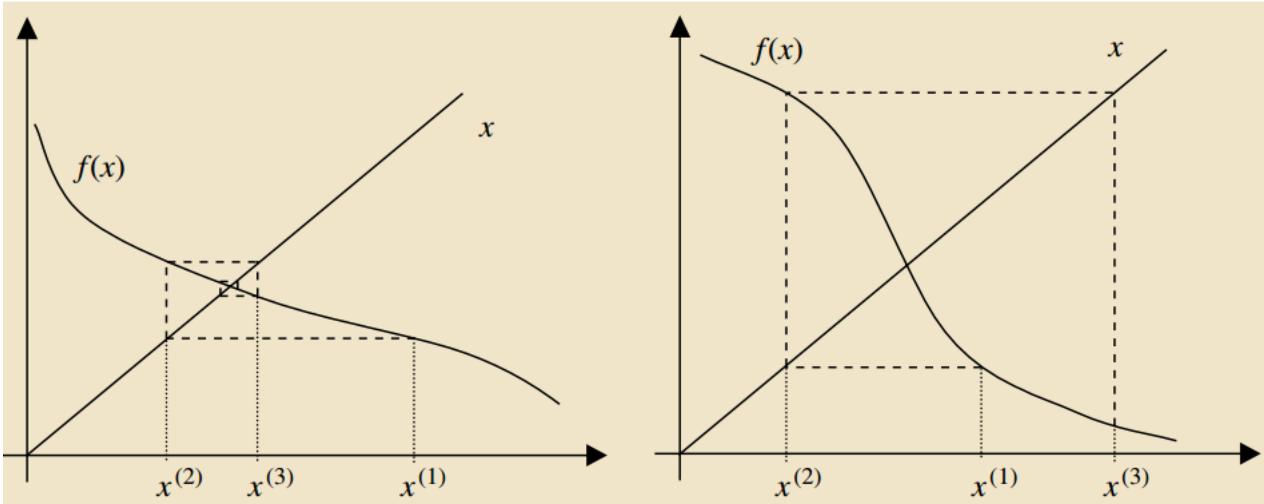

Figure 6: Converging (left) and diverging (right) iterations

Theorem 4.4 (contraction and NE)

If the best response mapping is a contraction on the entire strategy space, there is a unique NE in the game.

Theorem 4.4 connects contraction mapping with NE, though not clarifies which type of game satisfy the condition of contraction mapping. Suppose n players with strategy x_i and BR $x_i = f_i(x_{-i})$, then define A as

$$A = \begin{bmatrix} 0 & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & 0 & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & 0 \end{bmatrix}.$$

The spectral radius is defined as $\rho(A) = \{\max |\lambda| : Ax = \lambda x, x \neq 0\}$.

Theorem 4.5 (spectral radius rule)

The mapping $f(x) : R^n \to R^n$ *is a contraction if and only if* $\rho(A) < 1$ *everythere.*

Note on Problems *There are two problems of Theorem 4.5:*

- *Eigenvalue is not easy to calculate, however, it is enough to verify* $||A|| < 1$ *via the largest* *eigenvalue, that is, ensure that the sum of every row or column is smaller than 1.*

$$\sum_{i=1}^{n} |\frac{\partial f_k}{\partial x_i}| < 1 \quad or \quad \sum_{i=1}^{n} |\frac{\partial f_i}{\partial x_k}| < 1, \quad \forall k$$

In the case of two players, the condition can be rewritten as

$$|\frac{\partial f_1}{\partial x_2}| < 1 \quad and \quad |\frac{\partial f_2}{\partial x_1}| < 1$$

- *BR is not easy to find sometimes. However, via the implicit function theorem, Theorem 4.5 can be rewritten as* ***diagonal dominace****, that is, every elements on the diagonal should be greater than the sum of other elements in its row.*

$$\sum_{i=1, i \neq k}^{n} |\frac{\partial^2 \pi_k}{\partial x_k \partial x_i}| < |\frac{\partial^2 \pi_k}{\partial x_k^2}|, \quad \forall k$$

4.3.2 Univalent Mapping Argument

Theorem 4.6 (univalent and unique)

Suppose the strategy space of the game is convex and all equilibria are interior. Then, if the determinant $|H|$ is negative quasidefinite (i.e. if the matrix $H + H^T$ is negative definite) on the players' strategy set, there is a unique NE.

Note on *Note that the condition of univalent mapping argument is weaker than that of argument. In the case of two players, this theorem can be rewritten as*

$$|\frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} + \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2}| \leq 2\sqrt{\frac{\partial^2 \pi_1}{\partial x_1^2}\frac{\partial^2 \pi_2}{\partial x_2^2}}, \quad \forall x_1, x_2.$$

4.3.3 Index Theory Approach

Theorem 4.7 (index-theory and unique)

Suppose the strategy space of the game is convex and all payoff functions are quasiconcave. Then, if $(-1)^n|H|$ is positive whenever $\frac{\partial \pi_i}{\partial x_i} = 0$, all i, there is a unique NE.

Note on *Theorem 4.7 is also weaker than Theorem 4.4. However, Theorem 4.7 only requires the condition holds at equilibrium. In case of two players, Theorem 4.7 can be rewritten as*

$$\begin{vmatrix} \frac{\partial^2 \pi_1}{\partial x_1^2} & \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi_2}{\partial x_2^2} \end{vmatrix} > 0, \quad \forall x_1, x_2 : \frac{\partial \pi_1}{\partial x_1} = 0, \frac{\partial \pi_2}{\partial x_2} = 0$$

This can also be interpreted as that the product of the slopes of BR cannot be greater than 1, i.e.,

$$\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} < 1 \quad at x_1^*, x_2^*$$

4.4 Other properties of NE: Pareto optimality, Stability and Interiority

Definition 4.3 (Pareto frontier set)

The set of strategies such that each player can be made better off only if some other player is made worse off.

Definition 4.4 (Pareto optimal and Pareto inferior)

A set of strategies is Pareto optimal if they are on the Pareto frontier, otherwise, a set of strategies is Pareto inferior.

Definition 4.5 (stable equilibrium)

An equilibrium is considered stable (for simplicity we will consider asymptotic stability only) if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable.

The sufficient condition for asymptotically stable is

$$|\frac{dr_1}{dq_2}||\frac{dr_2}{dq_1}| < 1 \quad or \quad \frac{\partial^2 u_1}{\partial q_1 \partial q_2}\frac{\partial^2 u_2}{\partial q_1 \partial q_2} < \frac{\partial^2 u_1}{\partial q_1^2}\frac{\partial^2 u_2}{\partial q_2^2}$$

Note on *For example, B, C and D are all intersects of BR, and thus all NE. However, C is not stable, unless the status is C at the beginning, otherwise the status will move to B and D.*

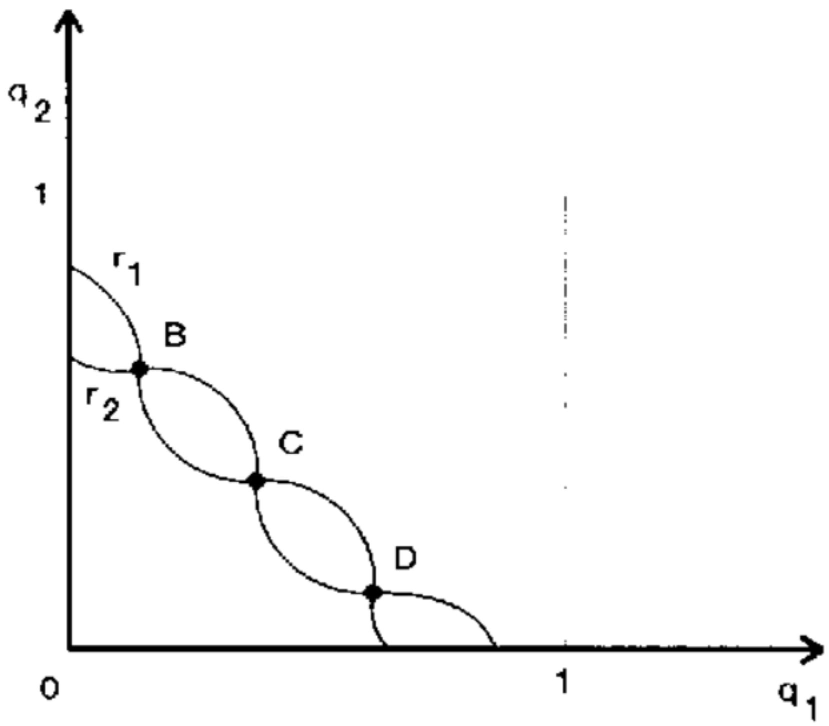

Figure 7: Example of stable and unstable NE

Definition 4.6 (Interior equilibrium and Boundary equilibrium)

Interior equilibrium is the one in which first-order conditions hold for each player. The alternative is boundary equilibrium in which at least one of the players select the strategy on the boundary of his strategy space.

5 Examples of finding NE

5.1 Hotelling Model

Consumers are uniformly distributed on $[0, 1]$ and each consumer buy one product from the closest vendor (minimizing transportation cost). Two vendors choose their locations simultaneously. In the unique Nash equilibrium, both vendors choose the midpoint of the boardwalk.

Note on Product differentiation *In the competition for market share, product differentiation is minimized in equilibrium. Similarly, in electoral competitions under bipartisan system, candidates tend to propose similar policies or express similar views on sensitive issues.*

Note on Multiple vendors *No pure NE is found currently.*

Note on Real examples *This model can be used to capture many real examples: 1 tax rate; 2 coco percent of chocolate bar. Actually, location means market share here.*

5.2 Cournot Model (Quantity Competition)

Two firms produce an identical product, and they set output levels (q_1, q_2) simultaneoulsy. Total output is $Q = q_1 + q_2$, and the market clearing price is $P = P(Q)$, each firm's utility is $u_i(q_1, q_2) = q_i P(Q) - C_i(q_i)$. Here we consider a special case where the inverse market demand is linear $p(Q) = a - bQ$, and the cost functions are also linear, $c(q_i) = c_i q_i$. This game thus can solve by BR equations via FOC. For example,

$$\beta_1(q_2) = \begin{cases} \frac{a-c_1}{2b} - \frac{q_2}{2} & q_2 \leq \frac{a-c_1}{b} \\ 0 & q_2 > \frac{a-c_1}{b} \end{cases}.$$

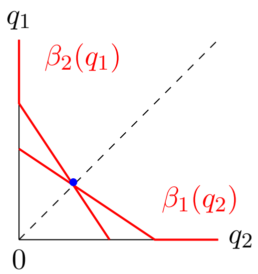

Figure 8: Case: $c_1 = c_2$

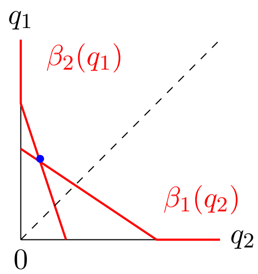

Figure 9: Case: $c_1 < c_2$

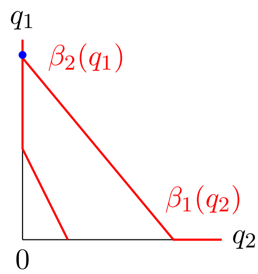

Figure 10: Corner Solution

IESDS. Cournot model can also be solved by iterated deletion of strictly dominated strategies. We first define firm i's best response function $r_i : [0, \infty) \to [0, \infty)$, and by FOC we have $r(q_{-i}) = \frac{a-c}{2b} - \frac{q_{-i}}{2}$. At the beginning of iteration, our strategy space is $S_i^0 = [0, \infty)$. In the first period, any strategy above $r(0)$ is strictly dominated for each firm, then the updated strategy space is $S_i^1 = [0, r(0)]$. Similarly in the second period, any strategy below $r(r(0)) = r^2(0)$ is strictly dominated, and our updated strategy space is $S_i^2 = [r^2(0), r(0)]$. Following this track, it converge to $\frac{a-c}{3b}$.

$$r^n(0) = -\frac{a-c}{b} \sum_{k=1}^{n} (-\frac{1}{2})^k$$

Multiple firms. When extending to n firms with linear inverse market demand, the NE is $q_i^* = \frac{a-c}{b(n+1)}$.

Corner Solution (Felix, 2022, Lec. 5). For example, in Figure 10, if $\frac{a-c_2}{b} < \frac{a-c_1}{2b}$, i.e., the vertical intercept of $\beta_1(q_2)$ is greater than the horizontal intercept of $\beta_2(q_1)$, then we have a corner solution.

Substitution and Complementation. There are three firms, A, B and C, in the market, in which A and B provides product 1, C provides product 2. $d \in [-1, 1]$ denotes the relationship between product 1 and 2, where $d = 0$ means no relationship, $d = 1$ means complement and $d = -1$ means substitute. The margin cost are c_A, c_B, c_C, and each firm decides the quantity q_A, q_B, q_C, the price function are

$$p_i = A - q_i - dq_j, \quad i, j = 1, 2$$

By FOC, we have,

$$\begin{aligned} q_A^*(d) &= \frac{1}{2}(c_B - c_A) + \frac{2A - c_A - c_B - d(A - c_C)}{6 - 2d^2} \\ q_B^*(d) &= \frac{1}{2}(c_A - c_B) + \frac{2A - c_A - c_B - d(A - c_C)}{6 - 2d^2} \\ q_C^*(d) &= \frac{3(A - c_C) - d(2A - c_A - c_B)}{6 - 2d^2} \end{aligned}$$

5.3 Bertrand Model (Price Competition)

Firm 1 and 2 choose price p_1 and p_2, their demand functions are follows, where $b > 0$ reflects substitution. There are not fixed costs of production, and marginal costs are constant at $c < a$. Finally we have $p_1^* = p_2^* = \frac{a+c}{2-b}$.

$$q_i(p_i, p_j) = a - p_i + bp_j$$

5.4 The Tragedy of the Commons

n farmers in a village chooses the number g_i of goats. Assume that the margin costs for each goat is c, and the value of each goat is $v(G)$, where $G = g_1 + ... + g_n$ is the total number of goats. Due to the limit of land resource, there exists $G_{max} : v(G) > 0 \quad \forall G < G_{max}, v(G) = 0 \quad \forall G \geq G_{max}$, and $v'(G) < 0, v''(G) < 0 \quad \forall G < G_{max}$. The payoff function is

$$g_i v(g_1 + \cdots + g_{i-1} + g_i + g_{i+1} + \cdots + g_n) - cg_i.$$

Let G^* denotes the total number in equilibrium when they make separate decisions, and summarizing all FOC conditions, we have

$$v\left(G^*\right)+\frac{1}{n} G^* v^{\prime}\left(G^*\right)-c=0.$$

However, the optimal quantity for the whole society $G^E < G^*$.

$$v\left(G^E\right)+G^E v^{\prime}\left(G^E\right)-c=0$$

5.5 A Model of Sales (Varian, 1980)

There are N firms and two types of consumers (informed I and uniformed U) in the market, and the population of consumers is 1 ($I+U=1$). Saying that the willingness to pay v is the same for all consumers, and firms do bertrand competition with marginal cost c. The payoff function for firm i given charging $p_i \in [c, v]$ is the following, where k is the number of firms charging the lowest price.

$$\pi_i\left(p_i, p_{-i}\right)= \begin{cases}\left(p_i-c\right) \frac{U}{N} & \text { if } p_i>\min_{j \neq i} p_j \\ \left(p_i-c\right)\left(\frac{U}{N}+\frac{l}{k}\right) & \text { if } p_i=\min_{j \neq i} p_j\end{cases}$$

Lemma 5.1

There is no pure Nash equilibrium in this game, and a unique symmetric equilibrium such that

$$\begin{aligned}
&\bar{p}(F)=v \\
&(\underline{p}(F)-c)\left(\frac{U}{N}+I\right)=(v-c) \frac{U}{N} \\
&(p-c)\left[\frac{U}{N}+(1-F(p))^{n-1} I\right]=(v-c) \frac{U}{N}, \forall p \in[\underline{p}(F), \bar{p}(F)]
\end{aligned}$$

Here $F(p), f$ means the cdf,pdf of strategy p and $\bar{p}=\inf\{p \mid F(p)=1\}$, $\underline{p}=\sup\{p \mid F(p)=0\}$. This MNE means

1. *The highest possible price is v.*
2. *The lowest price give the same profit for highest price.*
3. *Charging any price in between gives the same profit.*

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